Directed graph algorithms

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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Directed graphs

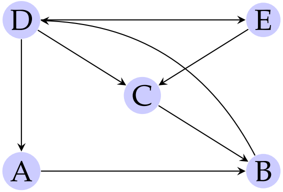
- Directed graphs are graphs with directed edges
- Some operations are more meaningful or challenging in directed graphs
- We will cover some of these operations, and some interesting sub-types of directed graphs
  - Transitive closure
  - Directed acyclic graphs
  - Topological ordering

Some terminology

- For any pair of nodes u and v in a directed graph
  - A directed graph is *strongly connected* if there is a directed path between u to v and v to u
  - A directed graph is *semi-connected* if there is a directed path between u to v or v to u
  - A directed graph is *weakly connected* if the undirected graph obtained by replacing all edges with undirected edges result in a connected graph

Checking strong connectivity

- Naive attempt: traverse the graph independently from each node (strongly connected if all traversals visit all nodes)



Checking strong connectivity

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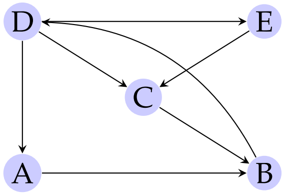
  - Time complexity:



Checking strong connectivity

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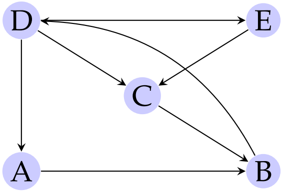
  - Time complexity: $O(n(n+m))$



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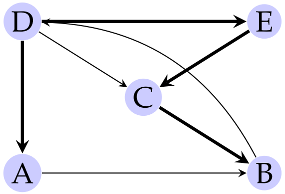
- A better one:
  - traverse the graph from an arbitrary node



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- Naive attempt: traverse the graph independently from each node (strongly connected if all traversals visit all nodes)
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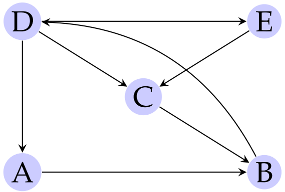

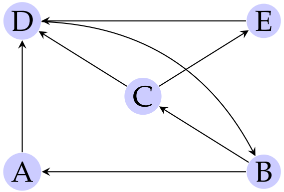

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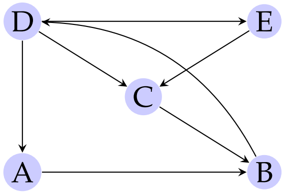

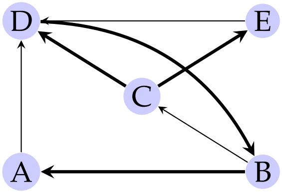

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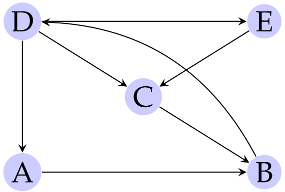

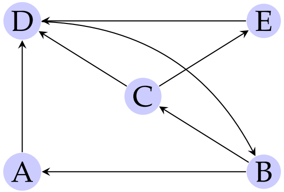

Checking strong connectivity

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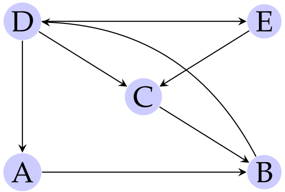
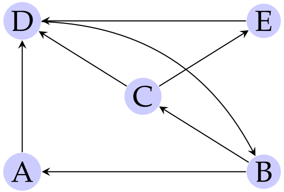
- A better one:
  - traverse the graph from an arbitrary node
  - reverse all edges, traverse again
  - intuition: if there is a reverse path from D to A, then D is reachable from A
- Time complexity: $O(n + m)$
- Note: we do not need to copy the graph, we only need to do 'reverse edge' queries





Transitive closure

- We know that graph traversals answer reachability questions about two nodes efficiently
- Pre-computing all nodes reachable from every other node is beneficial in some applications
- The *transitive closure* of a graph is another graph where
  - The set of nodes are the same as the original graph
  - There is an edge between two nodes u and v if v is reachable from u
- For an undirected graph, transitive closure can be computed by computing the connected components

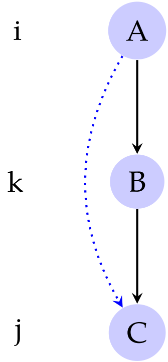
Computing transitive closure on directed graphs

- A straightforward algorithm:
  - run n graph traversals, from each node in the graph,
  - add an edge between the start node to any node discovered by the traversal
  - time complexity is $O(n(n+m))$
- Floyd-Warshall algorithm is another well-known algorithm that runs more efficiently in some settings

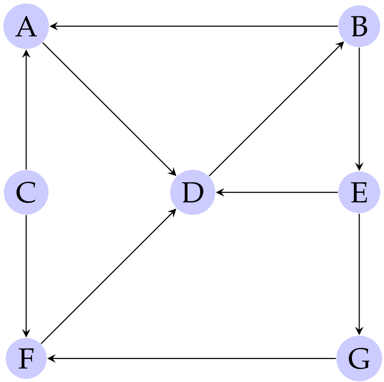
Floyd-Warshall algorithm

for finding transitive closure

- Remember that transitive closure of a graph is another graph
- Floyd-Warshall algorithm is an iterative algorithm that computes the transitive closure in n iterations
- The algorithm starts with setting transitive closure to the original graph
- For $k = 1 \dots n$
  - Add a directed edge (v_i, v_j) to transitive closure if it already contains both (v_i, v_k) and (v_k, v_j)
- It is efficient if graph is implemented with an adjacency matrix and it is not sparse



Floyd-Warshall demonstration



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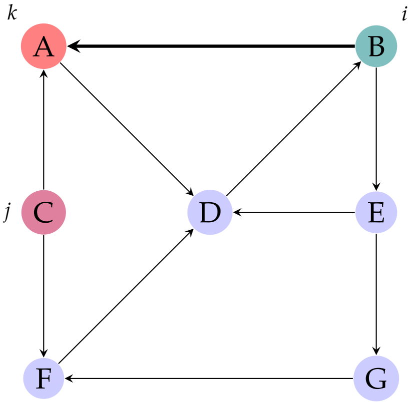

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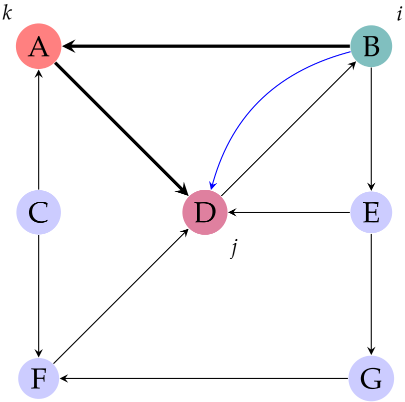

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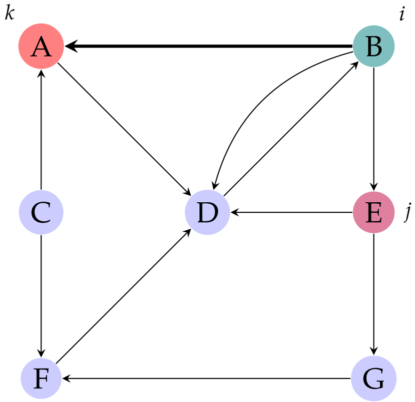

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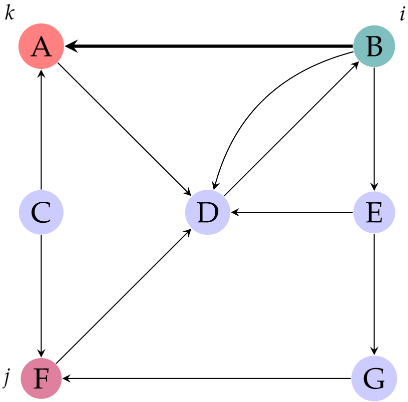

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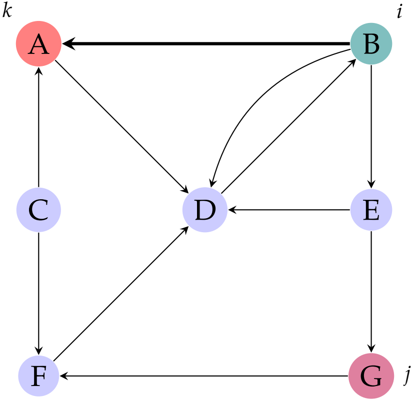

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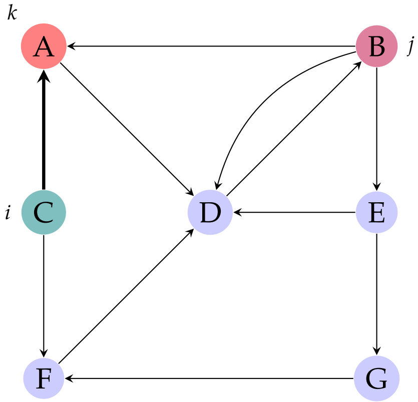

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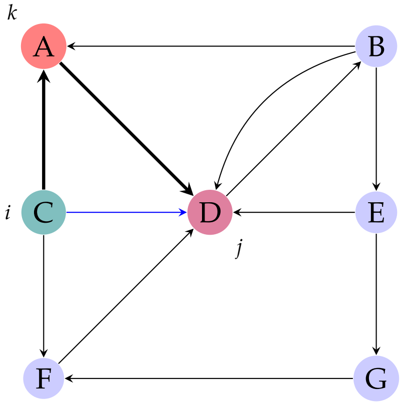

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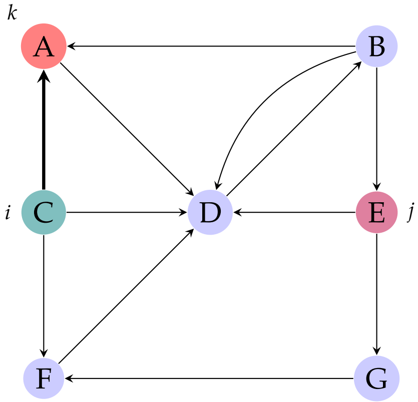

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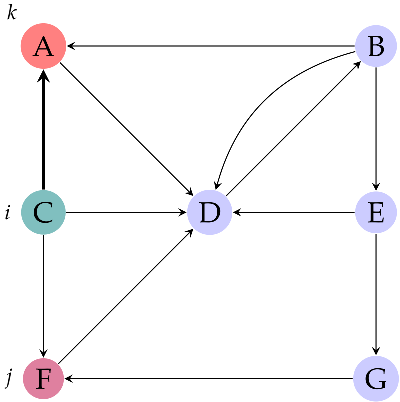

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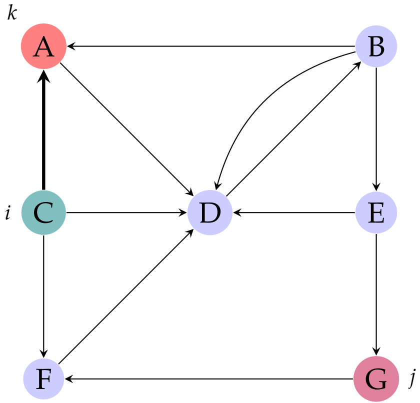

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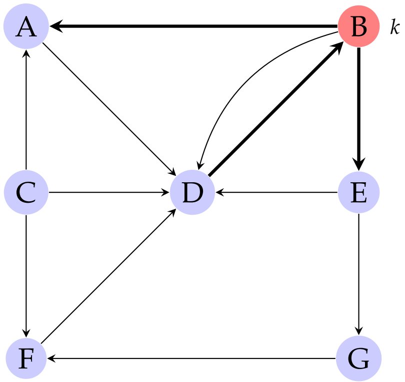

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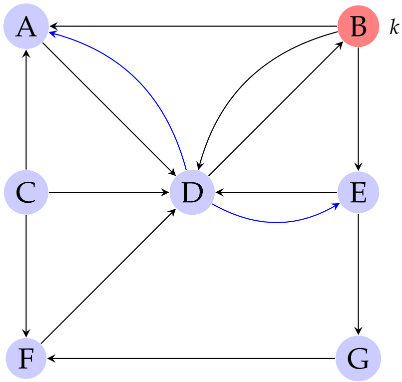

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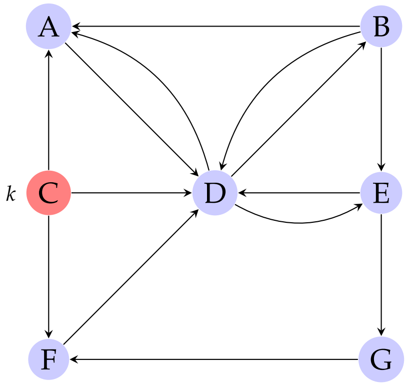

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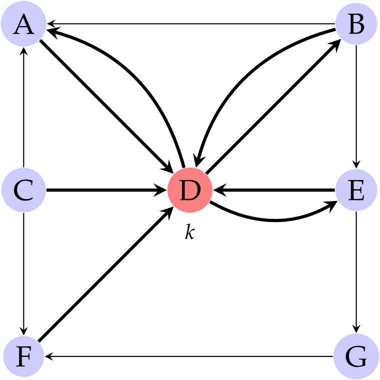

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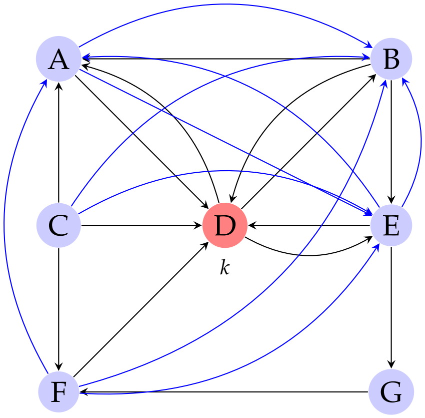

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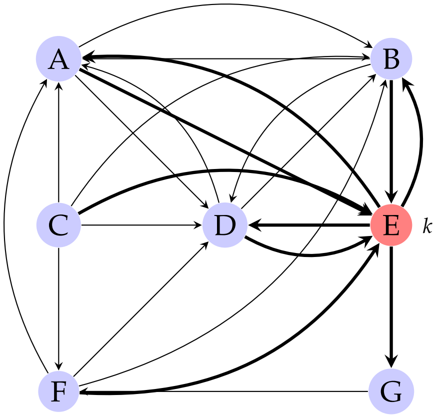

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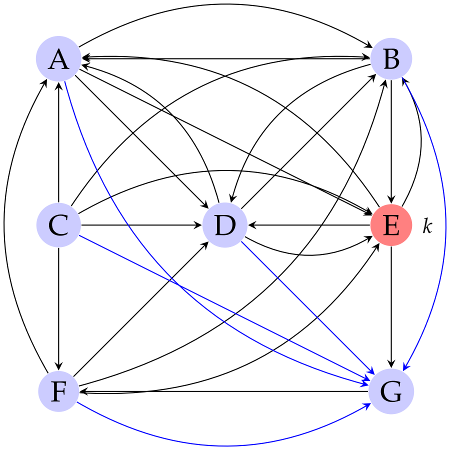

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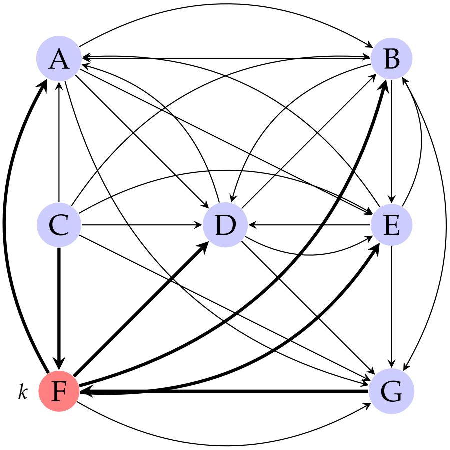
Floyd-Warshall demonstration



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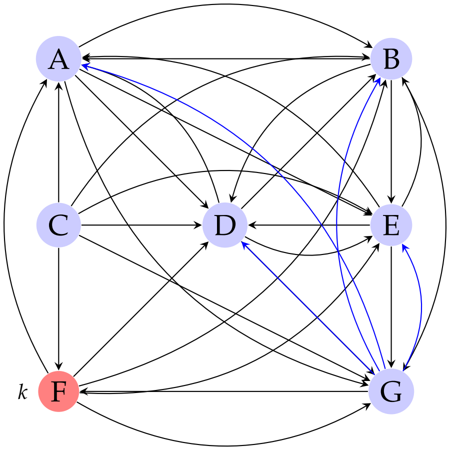
Floyd-Warshall demonstration



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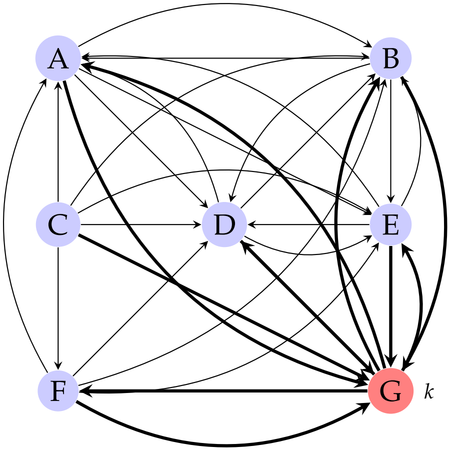

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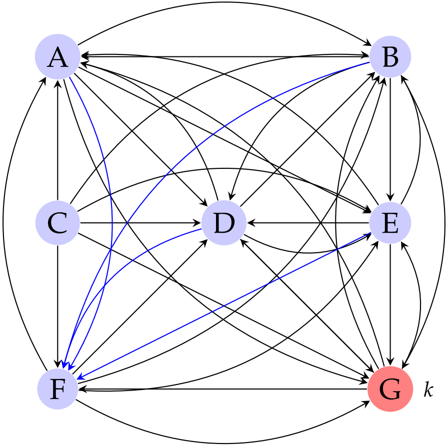

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Floyd-Warshall algorithm

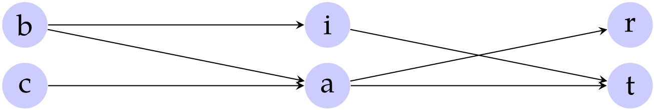
adjacency matrix implementation

```
T = [row[:] for row in G]
for k in range(n):
  for i in range(n):
    if i == k: continue
    for j in range(n):
      if j == i or  j == k:
        continue
      T[i][j] = T[i][j] or \
                T[i][k] and T[k][j]
```

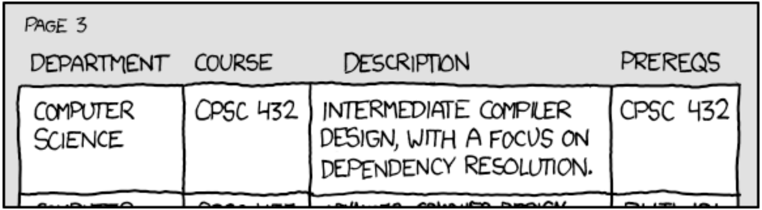
- Time complexity is $O(n^3)$
- Compare with repeated traversal: $O(n(n + m))$
  - Note that in a dense graph m is $O(n^2)$
- A version of this algorithm is also used for finding shortest paths in weighted graphs (later in the course)

Directed acyclic graphs

- *Directed acyclic graphs* (DAGs) are directed graphs without cycles
- DAGs have many practical applications (mainly, dependency graphs)
  - Prerequisites between courses in a study program
  - Class inheritance in an object-oriented program
  - Scheduling constraints over tasks in a project
  - Dependency parser output (generally trees, but can also be more general DAGs)
  - A compact representation of a list of words:



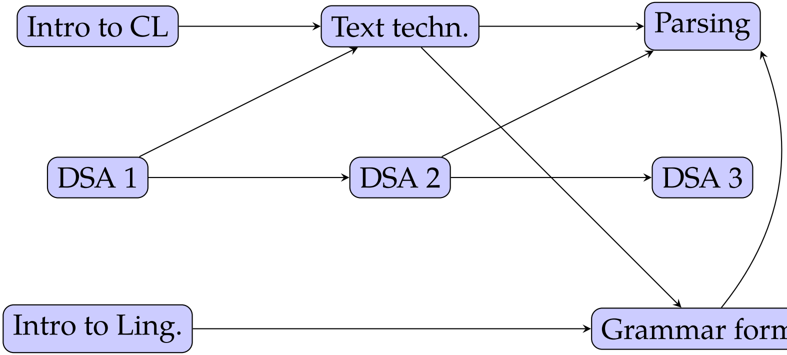
Directed acyclic graphs



https://www.xkcd.com/754/

DAG exammple

a (hypothetical) course prerequisite graph

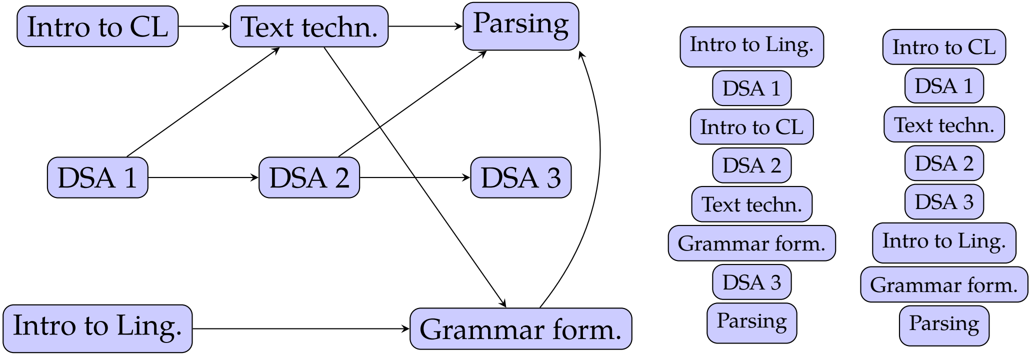


Topological order

- A *topological ordering* of a directed graph is a sequence of nodes such that for every directed edge (u, v) u is listed before v
- A topological ordering lists 'prerequisites' of a node before listing the node itself
- There may be multiple topological orderings
- In the course prerequisite example, a topological ordering lists any acceptable order that the courses can be taken

Topological order example

course prerequisites – two alternative topological orders



Topological sort

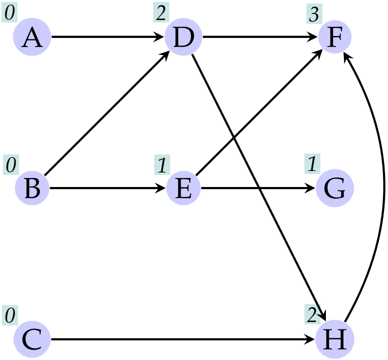
algorithm

```
topo, ready = [], []
incount = {}
for u in nodes:
   incount[u] = u.indegree()
   if incount[u] == 0:
       ready.append(u)
while len(ready) > 0:
    u = ready.pop()
    topo.append(u)
    for v in u.neighbors():
        incount[v] -= 1
        if incount[v] == 0:
            ready.append(v)
```

- Keep record of number of incoming edges
- A node is ready to be placed in the sorted list if there no unprocessed incoming edges
- Running time is $O(n + m)$
- If the topological ordering does not contain all the edges, the graph includes a cycle

Topological sort

demonstration

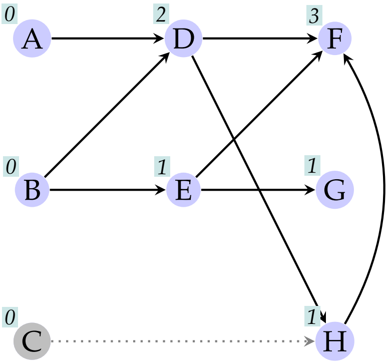


ready
A
B
C

sorted

Topological sort

demonstration



ready

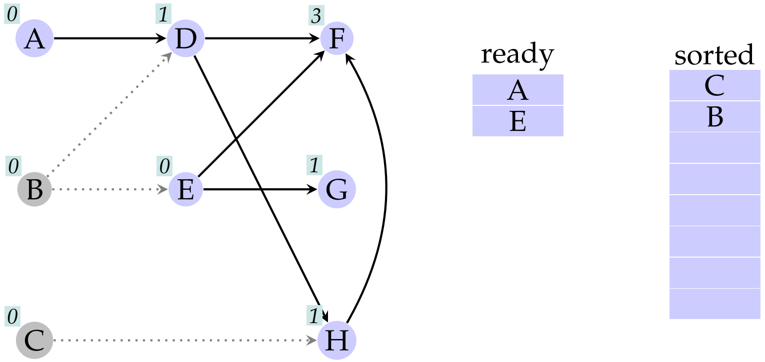
A
B

sorted

C

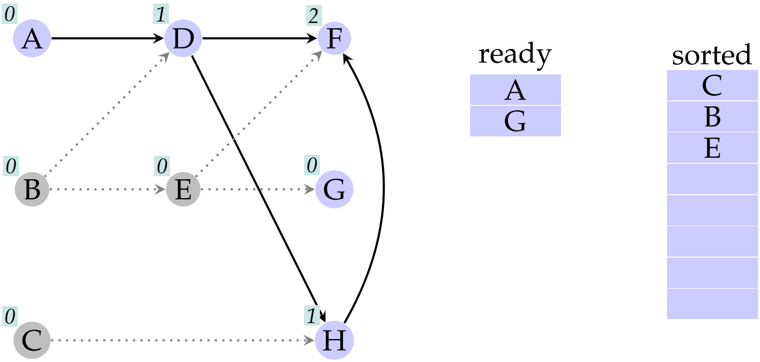
Topological sort

demonstration



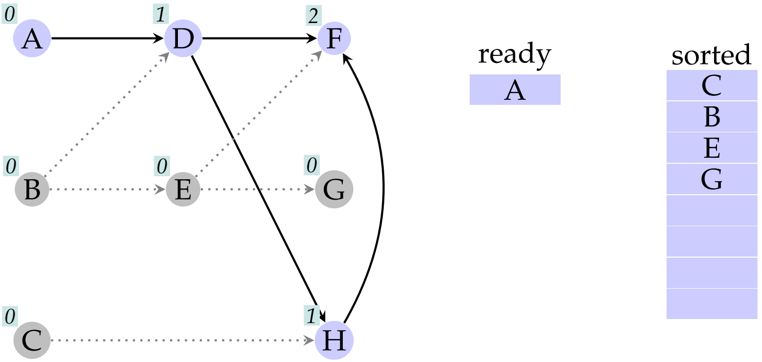
Topological sort

demonstration



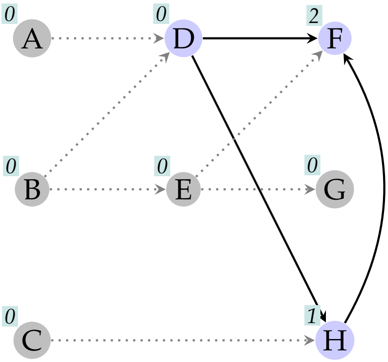
Topological sort

demonstration



Topological sort

demonstration



ready

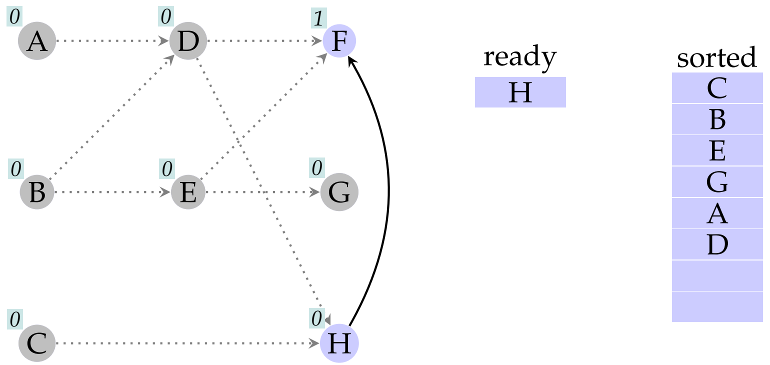
D

sorted

C
B
E
G
A

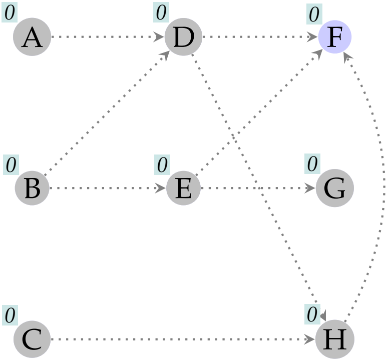
Topological sort

demonstration



Topological sort

demonstration



ready

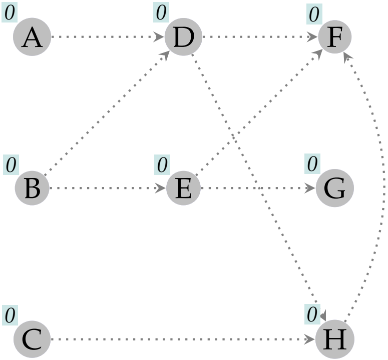
F

sorted

C
B
E
G
A
D
H

Topological sort

demonstration



ready

sorted

sorted
C
B
E
G
A
D
H
F

Summary

- Some operations on directed graphs are more challenging
- We covered
  - Finding strongly connected components
  - Finding the transitive closure of a digraph
  - DAGs and topological ordering
- Reading on graphs: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Next:

- More on graphs: shortest paths, minimum spanning trees

Acknowledgments, credits, references

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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